

Classical Euclidean general relativity from “left-handed area = right-handed area”

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Abstract

A classical continuum theory corresponding to Barrett and Crane’s model of Euclidean quantum gravity is presented. The fields in this classical theory are those of $SO(4)$ BF theory, a simple topological theory of an $so(4)$ valued 2-form field, $B_{\mu\nu}^{IJ}$, and an $so(4)$ connection. The left handed (self-dual) and right handed (anti-self-dual) components of B define a left handed and a, generally distinct, right handed area for each spacetime 2-surface. The theory being presented is obtained by adding to the BF action a Lagrange multiplier term that enforces the constraint that the left handed and the right handed areas be equal. It is shown that Euclidean general relativity (GR) forms a sector of the resulting theory. The remaining three sectors of the theory are also characterized and it is shown that, except in special cases, GR canonical initial data is sufficient to specify the GR sector as well as a specific solution within this sector.

Finally, the path integral quantization of the theory is discussed at a formal level and a heuristic argument is given suggesting that in the semiclassical limit the path integral is dominated by solutions in one of the non-GR sectors, which would mean that the theory quantized in this way is not a quantization of GR.

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1 Introduction

The Palatini action for general relativity (GR) may be written in the form¹

$$I_{GR} = \int \epsilon_{IJKL} e^I \wedge e^J \wedge R^{KL}, \quad (3)$$

where e_μ^I is the vierbein, $R_{\mu\nu}^{KL}$ is the curvature of the spin connection, and the capital latin indices $I, J, \dots \in \{0, 1, 2, 3\}$ are vector indices under the frame rotation group - $SO(4)$ in the case of Euclidean GR which we shall consider.

(3) is just the $SO(4)$ BF² action, $\int B_{KL} \wedge R^{KL}$ with the restriction that

$$B_{KL} = \epsilon_{IJKL} e^I \wedge e^J. \quad (4)$$

Barrett and Crane [BaCr97] have proposed a model of simplicial Euclidean quantum gravity in which the amplitude for any history is its amplitude in the simplicial BF theory of Ooguri [Oog92],³ but the histories are restricted by constraints designed to enforce a simplicial version of (4). In fact these constraints have several branches of solutions besides those corresponding to (4) [Baez97].

It turns out [Urb83, CDJM91, Ben95, Rei95] that when (4) holds the geometrical (gauge invariant) information contained in the vierbein e , is equivalent to the gauge invariant information in the self-dual, or “left handed”, part, B^+ , of B , as well as that in the anti-self-dual, or “right-handed”, part

¹ The wedge product of two forms, $a_{\alpha_1\dots\alpha_m}$ and $b_{\beta_1\dots\beta_n}$, is defined as

$$[a \wedge b]_{\alpha_1\dots\alpha_m\beta_1\dots\beta_n} = a_{[\alpha_1\dots\alpha_m} b_{\beta_1\dots\beta_n]}. \quad (1)$$

The integral over a d dimensional submanifold, s , of a d -form, f , living on a manifold M is defined as

$$\int_s f = \int f_{\alpha_1\dots\alpha_d} \frac{\partial x^{[\alpha_1}}{\partial \sigma^1} \dots \frac{\partial x^{\alpha_d]}}{\partial \sigma^d} d^d \sigma, \quad (2)$$

where $\sigma^1, \dots, \sigma^d$ are coordinates on s , and x^α are coordinates on M .

²The BF theory for (internal) gauge group G is defined by the action $\int tr[B \wedge F]$ when F is the curvature and B is a 2-form field taking values in the adjoint representation of the algebra of G , just like F . The trace is taken in this adjoint representation. See [Sch78], [Hor89].

³ Ooguri’s model was of $SU(2)$ BF theory. Here the (trivial) extension to $SO(4)$ BF theory is used. Barrett and Crane also consider regulating their model by q-deforming the $so(4)$ gauge algebra, which presumably corresponds to introducing a cosmological constant. In this case the Ooguri model is replaced by the Crane-Yetter model [BaCr97].

B^- . Since B^+ and B^- are well defined for any B they can be used to define a left handed and a right handed spacetime geometry whether (4) holds or not. Baez [Baez97] has pointed out that the Barrett-Crane constraints imply that the left and right geometries of 3-surfaces embedded in spacetime are equal. In particular this implies that the areas of spacetime 2-surfaces computed from B^+ and from B^- are equal. As the author hopes to elaborate in a future publication, the equality of self-dual and anti-self-dual areas is in fact the essence of the Barrett-Crane constraints.

Here we will examine the analogous *classical continuum* theory, defined by the stationary points of the BF action with respect to variations that preserve the equality of self-dual and anti-self-dual areas. We will find that a branch of these stationary points reproduce GR. An action for this theory is constructed in §2, and its several branches of solutions are classified and studied in section §3. Finally, §4 is a heuristic comment on the quantization of the theory, and §5 summarizes the conclusions.

2 Setting up the theory

Let's begin by briefly reviewing self-duality and the decomposition of $SO(4)$ into left and right handed factors. The dual of an antisymmetric tensor a^{IJ} is defined as⁴

$$a^{\star IJ} = \frac{1}{2}\epsilon^{IJ}_{KLA}a^{KL}. \quad (5)$$

Any antisymmetric tensor a^{IJ} can be split into a self-dual component a^+ satisfying $a^+ = a^{+\star}$, and an anti-self-dual component a^- satisfying $a^- = -a^{-\star}$, according to

$$a^{\pm} = \frac{1}{2}[a \pm a^{\star}]. \quad (6)$$

$SO(4)$ factors into a “left-handed” $SU(2)$ and a “right-handed” $SU(2)$:

$$SO(4) = SU(2)_L \otimes SU(2)_R. \quad (7)$$

Self-dual tensors transform non-trivially only under $SU(2)_L$. Under $SU(2)_R$ they are constant. For this reason they are also called “left-handed” tensors. Similarly anti-self-dual tensors transform non-trivially only under $SU(2)_R$,

⁴ $SO(4)$ tensor indices are raised with the $SO(4)$ metric δ^{IJ} , i.e. the Kronecker delta.

and are called “right-handed” tensors. (Anti-)self-dual tensors have only three independent components, which will be taken to be

$$a^{\pm i} = \pm 2a^{\pm 0i}. \quad (8)$$

(Lower case latin indices run over $\{1, 2, 3\}$). These components transform as adjoint vector ($= SO(3)$ vector) components, under $SU(2)_L$ in the case of a left-handed tensor, and under $SU(2)_R$ in the case of right-handed tensors.

It is easy to verify from (6) and (8) that for any two antisymmetric tensors a^{IJ} and b_{KL}

$$a^{IJ}b_{IJ} = a^{+i}b_i^+ + a^{-i}b_i^- . \quad (9)$$

It follows that the $SO(4)$ BF action, $\int B_{IJ} \wedge R^{IJ}$ can be written as the difference of a left handed and a right handed piece:

$$I_{BF} = 2 \int \Sigma_i^+ \wedge R^{+i} - \Sigma_i^- \wedge R^{-i} . \quad (10)$$

with $\Sigma^{\pm i} = 2B^{\pm 0i}$. This reduces to the Palatini action for GR when a condition equivalent to (4) is imposed:

$$\Sigma_i^\pm = \pm 2E^{\pm 0i} = \pm e^0 \wedge e^i + \frac{1}{2}\epsilon_{ijk}e^j \wedge e^k , \quad (11)$$

where $E^{IJ} = e^I \wedge e^J$. It follows from the factorization (7), or by simple algebra, that the left and right handed components, R^+ and R^- , of the $SO(4)$ curvature R are the curvatures of, A^+ and A^- , the $SU(2)_L$ and $SU(2)_R$ components, respectively, of the spin connection. Since any pair of $SU(2)$ connections define an $SO(4)$ connection via (7), there are no restrictions A^\pm . I_{BF} is thus the difference of two independent $SU(2)$ BF actions. GR is obtained by coupling the left and right handed fields via the constraint (11).

Perhaps surprisingly, when (11) holds both Σ^+ and Σ^- contain enough information to compute the geometry of spacetime. The metric can be determined from either Σ^+ or Σ^- [Urb83, CDJM91, Ben95, Rei95]. Our principal interest will be areas.

Just as the length of a curve in spacetime can be written in terms of the usual metric, the area of a 2-surface can be written in terms of an “area metric”:

$$\text{Area}(\sigma) = \int_\sigma \sqrt{m_{\alpha\beta\gamma\delta} t^{\alpha\beta} t^{\gamma\delta}} du^1 du^2 , \quad (12)$$

where σ is any smooth 2-surface, (u^1, u^2) are coordinates on σ , $t^{\alpha\beta} = 2\frac{\partial x^{[\alpha}}{\partial u^1}\frac{\partial x^{\beta]}}{\partial u^2}$ is the tangent bivector of σ (the x^γ are spacetime coordinates), and

$$m = \frac{1}{2}E^{IJ} \otimes E_{IJ} \quad (13)$$

is the area metric.

When (11) holds this area metric can be written in terms of either Σ^+ or Σ^- . It is equal to both the left area metric, m^+ , and the right area metric, m^- , defined by

$$m^\pm = \Sigma^{\pm i} \otimes \Sigma_i^\pm - \frac{1}{4}V^\pm, \quad (14)$$

with

$$V^\pm = \pm 4\Sigma^{\pm i} \wedge \Sigma_i^\pm. \quad (15)$$

V^+ and V^- both equal the 4-volume form $V = e^0 \wedge e^1 \wedge e^2 \wedge e^3$.

If Σ^+ and Σ^- are not required to satisfy (11) the left and right areas and 4-volumes are still defined, though they need not agree. m^+ and V^+ define a “left” geometry, while m^- and V^- defines a, generally distinct, “right” geometry.⁵

Suppose the left and right areas are *required* to be equal. What does this tell us about the form of Σ^\pm ? It turns out that when the average 4-volume form $\frac{1}{2}(V^+ + V^-) \neq 0$ the equation $m^+ = m^-$ has four isolated branches of solutions⁶ one of which is the vierbein form (11). In addition there are two branches of solutions with $\frac{1}{2}(V^+ + V^-) = 0$.

Thus standard GR can be obtained as a sector of the solutions of the theory defined by

$$I_\psi = \int \Sigma_i^+ \wedge F^{+i} - \Sigma_i^- \wedge F^{-i} \quad (17)$$

$$+ \int \psi^{\alpha\beta\gamma\delta} [\Sigma_{i\alpha\beta}^+ \Sigma_{\gamma\delta}^{+i} - \Sigma_{i\alpha\beta}^- \Sigma_{\gamma\delta}^{-i}] d^4x, \quad (18)$$

⁵One can also define left and right metrics for lengths from Σ^+ and Σ^- via Urbantke’s formula [Urb83, Ben95]

$$[\sqrt{g}g_{\mu\nu}]^\pm = \pm[\epsilon^{\alpha\beta\gamma\delta}\Sigma_{\mu\alpha}^1 \Sigma_{\beta\gamma}^2 \Sigma_{\delta\nu}^3]^\pm. \quad (16)$$

However the areas defined by these length metrics will generally be different from those defined by the left and right area metrics used here.

⁶The corresponding four branches of solutions to Barrett and Crane’s essentially equivalent equation on left and right handed area bivectors of the the 2-simplices of a 4-simplex, are already given in [Baez97].

where Σ^+ , and Σ^- are independent triplets of 2-form fields, F^+ and F^- are the curvatures of two independent $SU(2)$ connections, and ψ is a Lagrange multiplier of density weight one, which satisfies $\psi^{\alpha\beta\gamma\delta} = \psi^{\gamma\delta\alpha\beta} = -\psi^{\beta\alpha\gamma\delta}$ and $\psi^{[\alpha\beta\gamma\delta]} = 0$. The last condition ensures that stationarity with respect to ψ requires only the equality of the left and right area metrics, $m_{\alpha\beta\gamma\delta}^\pm$, which have no totally antisymmetric part, and leaves the totally antisymmetric parts of $\Sigma^\pm \otimes \Sigma^\pm$, which form V^\pm , unconstrained.

The field equations implied by the stationarity of I_ψ are

$$0 = D^\pm \wedge \Sigma^\pm \quad (19)$$

$$0 = F_{\mu\nu}^{\pm i} + \frac{1}{4}\epsilon_{\mu\nu\alpha\beta}\psi^{\alpha\beta\gamma\delta}\Sigma_{\gamma\delta}^{\pm i} \quad (20)$$

$$0 = m^+ - m^- \quad (21)$$

$m^+ - m^-$ has 20 linearly independent components, so $m^+ - m^- = 0$ seems to provide just the right number of constraints to reduce the 36 degrees of freedom of (Σ^+, Σ^-) to the 16 degrees of freedom of the vierbein e_μ^I . A more careful study yields

Theorem:

1) If $m^+ = m^-$ and $V^+ + V^- \neq 0$ then there exists a non-degenerate, real, vierbein e_μ^I and signs $s_1, s_2 \in \{1, -1\}$ such that

$$\Sigma_i^+ = s^+[e^0 \wedge e^i + \frac{1}{2}\epsilon_{ijk}e^j \wedge e^k] \quad (22)$$

$$\Sigma_i^- = s^-[-e^0 \wedge e^i + \frac{1}{2}\epsilon_{ijk}e^j \wedge e^k] \quad (23)$$

$$V^+ = V^- = e^0 \wedge e^1 \wedge e^2 \wedge e^3. \quad (24)$$

2) If $m^+ = m^-$ and $V^+ + V^- = 0$ then there exists $T \in SO(3)$ and $s \in \{1, -1\}$ such that

$$\Sigma_j^- = sT_j^i\Sigma_i^+. \quad (25)$$

Proof: When $m^+ - m^- = 0$

$$\Sigma_i^+ \otimes \Sigma^{+i} - \Sigma_i^- \otimes \Sigma^{-i} = 4(V^+ + V^-) \quad (26)$$

because only the totally antisymmetric part of the expression on the left is non-zero.

Now let's consider 2): $V^+ + V^- = 0$ so

$$\Sigma_i^+ \otimes \Sigma^{+i} = \Sigma_i^- \otimes \Sigma^{-i}. \quad (27)$$

The $\Sigma^{\pm i}$ define linear functions $v^{\pm i} = \Sigma_{\alpha\beta}^{\pm i} v^{\alpha\beta}$ on the space \mathcal{A} of rank two antisymmetric spacetime tensors $v^{\alpha\beta}$. In terms of these linear functions (27) becomes $v_i^+ v^{+i} = v_i^- v^{-i}$. Now, $w^{+i} = u^{+i}$ implies $0 = [w - u]_i^+ [w - u]^{+i} = [w - u]_i^- [w - u]^{-i}$. This, in turn, implies that $w^{-i} = u^{-i}$. Obviously the converse also holds, so v^{+i} is an invertible function of the v^{-i} . It is easy to see from the linearity of $v^{\pm i}$ that v^{+i} and v^{-i} are related by a linear transformation. Finally, $v_i^+ v^{+i} = v_i^- v^{-i}$ implies that this transformation is orthogonal:

$$\exists O \in O(3) \text{ such that } v^{+i} = O^i_j v^{-j}. \quad (28)$$

This equation holds for all $v \in \mathcal{A}$, so it actually holds for the $\Sigma^{\pm i}$. If the orthogonal transformation is written as a product $O = sT$ of $T \in SO(3)$ and a sign factor $s \in \{1, -1\}$ the result is

$$\Sigma_j^- = s T_j^i \Sigma_i^+. \quad (29)$$

Now let's prove part 1), which concerns the case $V^+ + V^- \neq 0$. The totally antisymmetric tensor

$$\Sigma_i^+ \otimes \Sigma^{+i} - \Sigma_i^- \otimes \Sigma^{-i} = 4(V^+ + V^-). \quad (30)$$

defines a metric on \mathcal{A} which is proportional to $\epsilon_{\alpha\beta\gamma\delta}$. This metric, like ϵ is non-degenerate with signature $1, 1, 1, -1, -1, -1$. In order that the left side of (30) reproduce this non-degenerate metric it is necessary that the six Σ_i^\pm be linearly independent. Consequently there exists a basis $\Lambda_i^{\pm\alpha\beta}$ of \mathcal{A} such that

$$\Sigma_{i\alpha\beta}^+ \Lambda_j^{+\alpha\beta} = \delta_{ij} \quad \Sigma_{i\alpha\beta}^- \Lambda_j^{+\alpha\beta} = 0 \quad (31)$$

$$\Sigma_{i\alpha\beta}^+ \Lambda_j^{-\alpha\beta} = 0 \quad \Sigma_{i\alpha\beta}^- \Lambda_j^{-\alpha\beta} = \delta_{ij}. \quad (32)$$

Contracting (30) with Λ_j^+ one sees that

$$\Sigma_{j\alpha\beta}^+ = \frac{1}{3!} (V^+ + V^-)_{\mu\nu\sigma\tau} \epsilon^{\mu\nu\sigma\tau} \epsilon_{\alpha\beta\gamma\delta} \Lambda_j^{+\gamma\delta}. \quad (33)$$

When this is substituted back into (31) one finds that

$$\Sigma_i^+ \wedge \Sigma_j^+ = \delta_{ij} \frac{2}{3} (V^+ + V^-). \quad (34)$$

It follows that

$$\Sigma_i^+ \wedge \Sigma_j^+ = \delta_{ij} \frac{1}{3} \Sigma_k^+ \wedge \Sigma^{+k} \quad (35)$$

and, from the definition $V^+ = \frac{1}{4} \Sigma_k^+ \wedge \Sigma^{+k}$, that $V^+ = \frac{1}{2} (V^+ + V^-) = V^-$.

By a well known result (35) and the fact that $\Sigma_k^+ \wedge \Sigma^{+k} = 2(V^+ + V^-) \neq 0$ imply that there exists a real vierbein \bar{e}_μ^I and $s^+ \in \{1, -1\}$ such that

$$\Sigma_i^+ = s^+ [\bar{e}^0 \wedge \bar{e}^i + \frac{1}{2} \epsilon_{ijk} \bar{e}^j \wedge \bar{e}^k] \quad (36)$$

and $\bar{e}^0 \wedge \bar{e}^1 \wedge \bar{e}^2 \wedge \bar{e}^3 = V^+ \neq 0$ [Isr68][Ple77][CDJM91] (see [Rei95] appendix B for a proof in more or less the present notation).

Using this vierbein we may define a basis of anti self-dual 2-forms: $\Sigma'_i^- = -\bar{e}^0 \wedge \bar{e}^i + \frac{1}{2} \epsilon_{ijk} \bar{e}^j \wedge \bar{e}^k$. Then $m'^- = m^+ = m^-$ and $V'^- = V^+ = V^-$, so

$$\Sigma_i^- \otimes \Sigma^{-i} - \Sigma'_i^- \otimes \Sigma'^{-i} = 0. \quad (37)$$

By 2), which is already proven, this implies that

$$\Sigma_i^- = s^- T_i^j \Sigma'_j^- \quad (38)$$

for some $T \in SO(3)$ and $s^- \in \{1, -1\}$. Now let $A = T \otimes \mathbf{1}$ be the $SO(4)$ transformation composed of left handed factor T and right handed factor $\mathbf{1}$. Then $e_\mu^I = A^I{}_J \bar{e}_\mu^J$ and (s^+, s^-) satisfy the equations (22), (23), and (24). ■

The solutions of $m^+ = m^-$ can be characterized as follows: Given *any* triplet of 2-forms Σ_i^+ , $m^+ = m^-$ can be satisfied by choosing $\Sigma_i^- = O_i{}^j \Sigma_j^+$ where $O \in O(3)$. For those special Σ_i^+ that satisfy $\Sigma_i^+ \wedge \Sigma_j^+ \propto \delta_{ij}$, with $\Sigma_k^+ \wedge \Sigma^{+k} \neq 0$, there are four additional branches of solutions (described in part 1) of the Theorem). In one of these branches of special solutions Σ^+ and Σ^- are of the vierbein form (11).

Now suppose we look for the stationary points of the action I_ψ by first restricting attention to field configurations that extremize the action with respect to ψ and then looking for stationary points among such fields. The

solutions we are looking for are then stationary points of a difference of two BF actions, $\int \Sigma_i^+ \wedge F^{+i} - \Sigma_i^+ \wedge F^{-i}$, under variations subject to the constraint $m^+ = m^-$. For each branch of solutions to $m^+ = m^-$ an action I_{branch} may be obtained by substituting the general solution in that branch into I_ψ . The stationary points of I_ψ are stationary points of one or another of the branch actions. Conversely, if the fields (Σ^+, Σ^-) of a stationary point of a branch action belong only to that branch, i.e. do not lie in the intersection of two branches of solutions to $m^+ = m^-$, then this stationary point is also a stationary point of I_ψ . Stationary points of I_ψ in the intersection of several branches must be stationary points of the actions of all these branches.

I_ψ gives rise to four inequivalent branch actions

$$I_{GR} = \int \epsilon_{IJKL} e^I \wedge e^J \wedge R^{KL} \quad (39)$$

$$I_1 = 2 \int e_I \wedge e_J \wedge R^{IJ} \quad (40)$$

$$I_2 = \int \Sigma_i \wedge [F^{+i} + F^{-i}] = \int \epsilon_{IJKL} a^I B^J \wedge R^{KL} \quad (41)$$

$$I_3 = \int \Sigma_i \wedge [F^{+i} - F^{-i}] = 2 \int a_I B_J \wedge R^{IJ}, \quad (42)$$

where R is an $SO(4)$ curvature, Σ_i is a triplet of 2-forms, a is an $SO(4)$ vector, and B is an $SO(4)$ vector and a 2-form.

I_{GR} corresponds to Σ^\pm of the vierbein form, which constitute the branch of solutions (22), (23) to $m^+ = m^-$ with $s^+ = s^- = 1$. R appearing in I_{GR} , and all the other branch actions, is the $SO(4)$ curvature defined by F^+ and F^- appearing in the general form, I_ψ , of the action. That is, F^+ and F^- are the left and right handed components of R .

The signs $s^+ = s^- = -1$ in (22) and (23) lead to the action $-I_{GR}$, which (in the absence of matter) is classically equivalent to I_{GR} . Opposite signs, $s^+ = -s^-$, produce the actions $\pm I_1$. Finally, Σ^\pm having $V^+ + V^- = 0$, given by (25), yield the actions I_2 and I_3 . The matrix $T \in SO(3)$ in (25) is absorbed in an adjustment of the gauge of the right handed connection A^- . Put another way, the separate $SU(2)$ gauge freedoms of A^+ and A^- are reduced to a single gauge $SU(2)$ by fixing $T = \mathbf{1}$. This leaves two possibilities: $s = 1$, corresponding to I_3 , and $s = -1$, corresponding to I_2 .

3 Solutions

The solutions of the theory defined by I_ψ are generally solutions to one of the four branch actions on all of spacetime. However, in special cases they consist of solutions belonging to different branches spliced together at transition hypersurfaces. In the following the forms of the solutions to the branch actions I_1 , I_2 , and I_3 are outlined, and the conditions under which I_1 , I_2 , and I_3 branch solutions can be spliced into a GR solution are studied. The canonical formulation of the theory defined by I_ψ is also outlined, and it is shown that, generically, the left and right handed canonical data of a GR solution does not have an alternative development into a solution of one of the other branches, so the left and right handed data together specify the GR branch as well as the solution within the GR branch. On the other hand, left handed GR data alone (such as Ashtekar's canonical variables), while sufficient to specify a solution within the GR branch, always has alternative developments in the I_1 and I_3 branches.

The theories defined by I_1 , I_2 , and I_3 are extremely simple. The stationary points of I_3 satisfy the field equations

$$F^{+i} = F^{-i} \quad (43)$$

$$0 = D^\pm \wedge \Sigma. \quad (44)$$

The most obvious class of solutions thus has $A^+ = A^-$ with A^+ and Σ chosen to be compatible in the sense that $D^+ \wedge \Sigma = 0$. Bengtsson [Ben95] has solved this compatibility condition and obtained an expression for the connection in terms of Σ , $\Sigma_i^{*\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}\Sigma_{i\gamma\delta}$, the inverse of the 3×3 matrix

$$m_{ij} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}\Sigma_{i\alpha\beta}\Sigma_{j\gamma\delta}, \quad (45)$$

and

$$t_{\alpha\beta\gamma ij} = 3\epsilon_{ijk}\partial_{[\alpha}\Sigma_{\beta\gamma]}^k. \quad (46)$$

His solution is

$$A_\mu^{+i} = -2t_{\mu\beta\gamma}{}^i{}_j\Sigma_k^{*\beta\gamma}m^{-1}{}^{jk} - 2\Sigma_{\mu\beta}^i\Sigma_j^{*\beta\gamma}t_{\gamma\delta\sigma}{}_{km}\Sigma_n^{*\delta\sigma}m^{-1}{}^{mn}m^{-1}{}^{jk}. \quad (47)$$

Of course this holds only provided that m is invertible, which it is for generic Σ . Thus for generic Σ there is always a corresponding solution and, since

the compatibility of the connection with Σ defines the connection completely, (44) implies that $A^+ = A^-$. There are also solutions for which $A^+ \neq A^-$, but in these both Σ and $F^+ = F^-$ must be algebraically special: If the matrix $n_{ij} = \epsilon^{\alpha\beta\gamma\delta} H_{i\alpha\beta} H_{j\gamma\delta}$ corresponding to any linear combination $H = a\Sigma + bF^+$ is invertible then (44) and the Bianchi identities imply that A^+ and A^- are given by the same expression in terms of H and are thus equal.

At stationary points of I_2 $F^+ = -F^-$. One immediately sees that the fields must be algebraically special in all the solutions: $D^- \wedge F^- = 0$ implies $D^- \wedge F^+ = 0$, just as for I_3 branch solutions. Thus, if F^+ , Σ , or any linear combination satisfy the invertibility condition then $A^+ = A^-$, implying that F^+ equals F^- instead of $-F^-$. That is, in this branch the curvature is zero, or the invertibility condition does not hold for any linear combination of the curl free fields F^+ and Σ . The most obvious class of solutions in this branch has $A^+ = -A^- = n^i A$, with n^i an $SU(2)$ adjoint vector field, and A a 1-form field. As in the I_3 branch, there are some additional solutions beyond these most obvious ones.

Stationarity of I_1 implies only one field equation:

$$D \wedge [e_I \wedge e_J] = 0 \quad (48)$$

This equation ensures that the connection ω is compatible with e in the sense that $D \wedge e = 0$.⁷ The other ostensible field equation, which follows from stationarity with respect to e , is

$$0 = e_J \wedge R^{IJ} = D \wedge D \wedge e^I, \quad (51)$$

and is implied by (48). Any e_μ^I and compatible connection ω_μ^{IJ} will thus satisfy the field equations.

While both GR and I_1 branch solutions can be described in terms of e and ω these fields are defined differently in terms of the fundamental variables

⁷ *Proof.* (48) can be expanded as

$$[D \wedge e_I] \wedge e_J - [D \wedge e_J] \wedge e_I = 0. \quad (49)$$

This implies $[D \wedge e_I] \wedge e_J \wedge e_I = 0 \ \forall J$. Thus $D \wedge e_I = K \wedge e_I$ for some 1-form K . Substituting this into (48) we find $0 = K \wedge e_I \wedge e_J$, which implies $K = 0$, and thus

$$D \wedge e_I = 0. \quad (50)$$

■

Σ^\pm, A^\pm in the two branches. In terms of the fundamental variables the GR branch can be distinguished from the I_1 branch using the fact that

$$\Sigma_{\mu[\alpha}^{+1} \Sigma_{\beta\gamma}^{+2} \Sigma_{\delta]\nu}^{+3} = \frac{s^-}{s^+} \Sigma_{\mu[\alpha}^{-1} \Sigma_{\beta\gamma}^{-2} \Sigma_{\delta]\nu}^{-3} \quad (52)$$

with $\frac{s^-}{s^+} = 1$ in the GR branch and -1 in the I_1 branch. Both sides of the equation are non-zero; $\Sigma_{\mu[\alpha}^{+1} \Sigma_{\beta\gamma}^{+2} \Sigma_{\delta]\nu}^{+3} = \frac{1}{8} V_{\alpha\beta\gamma\delta}^+ g_{\mu\nu}$, where $g_{\mu\nu} = e_\mu^I e_I \nu$ is the metric in either branch. The GR branch can of course also be distinguished from the I_2 and I_3 branches because, unlike in the GR branch, $V^+ + V^- = 0$ in these branches.

Under what circumstances can one splice together a GR solution and a solution of another branch? It turns out to be possible only in spacial cases. Evolution from the GR branch to the I_1 or I_2 branches cannot occur across a hypersurface on which the vierbein is well defined and non-degenerate ($\det[e_\mu^I] \neq 0$). Evolution from the GR branch to the I_3 branch can occur when the vierbein is non-degenerate, but requires a condition on the curvature on the hypersurface of transition, which doesn't hold on any hypersurface in a generic solution of GR.

To demonstrate these claims we need the junction conditions that spliced solutions must satisfy at the transition hypersurface. To derive these we first need to delimit the types of fields we are willing to consider as solutions to the field equations. Solutions will be required to be piecewise differentiable, with the set of points of non-differentiability a union of differentiable sub-manifolds. Moreover, at 3-surfaces of non-differentiability in the interior of the spacetime the fields are required to have well defined, finite limiting values when the 3-surface is approached from either side, and parallel transport along any differentiable curve is required to depend continuously on the curve. Requiring the points of non-differentiability to form differentiable manifolds makes it possible to replace the differential field equations by junction conditions at these manifolds. The requirement that the fields have well defined limits at 3-surfaces in the interior does not disallow the development of singularities in the fields (which would certainly exclude physically interesting solutions) but simply states that we will not develop solutions through such singularities. Of course singularities might develop on hypersurfaces that are do not cut the spacetime in two.⁸ The stationarity of the action will require

⁸Such surfaces can still be considered as excluded from the spacetime so that they do not form part of its interior.

some sort of junction condition to hold across these surfaces but we will not need these. Finally, allowing the parallel propagator along a curve to depend discontinuously on the curve would lead to ambiguous field equations in some cases.

Now suppose we slice an open region of spacetime with trivial topology into differentiable hypersurfaces H_t , continuously parametrized by a “time” $t \in \mathbf{R}$. Let’s also adopt adapted coordinates x^μ with $x^0 = t$, and let latin indices $a, b, c, \dots \in \{1, 2, 3\}$ so that they indicate “spatial” components in the tangent hyperplanes to H_t . It follows immediately from the restrictions placed on the connection that $A_a^{\pm i}$ is continuous in time, t , almost everywhere: The parallel propagator along a spatial curve, kept fixed in the coordinates x^a , is a continuous function of time. Moreover, our hypotheses imply that almost everywhere on H_t A^\pm has well defined limiting values as H_t is approached from either side. Thus the integrals of the two limits of the connection on H_t along any differentiable curve are equal, implying that the limits themselves are equal almost everywhere.

The other quantity that must be continuous in t (almost everywhere on H_t) is Σ_{iab}^\pm . This can be seen by considering the integrals of the field equations $D^\pm \wedge \Sigma^\pm = 0$ over infinitesimal three dimensional “Gaussian boxes”. For a connection of the form we are allowing it is possible to choose a gauge so that the connection vanishes at a point inside a given box. In this gauge the connection terms appearing in the covariant curls $D^\pm \wedge \Sigma^\pm$ can be neglected inside the box as it is shrunk to infinitesimal size, and Stoke’s theorem can be applied. Choosing suitable boxes it is then easy to show that the integrals of Σ^\pm over any spatial 2-surface which is fixed in the coordinates x^a is continuous in t , and the existence of the two one sided limits of Σ^\pm on H_t then implies that Σ_{iab}^\pm is continuous in t almost everywhere on H_t .

The continuity of $A_a^{\pm i}$ and Σ_{iab}^\pm , or equivalently $\tilde{E}_i^{\pm a} = \epsilon^{abc} \Sigma_{iab}^\pm$, can also be understood in terms of a canonical formulation of the theory. The action (18) can be rewritten with the spatial and time components explicitly separated:

$$\begin{aligned} I_\psi = & \int dt \int d^3x \dot{A}_a^{+i} \tilde{E}_i^{+a} - \dot{A}_a^{-i} \tilde{E}_i^{-a} + A_0^{+i} D_a^+ \tilde{E}_i^{+a} - A_0^{-i} D_a^- \tilde{E}_i^{-a} \\ & + 2\Sigma_{i0a}^+ \tilde{B}^{+ia} - 2\Sigma_{i0a}^- \tilde{B}^{-ia} + \tilde{\theta}^{ab} [\Sigma_{0a}^{+i} \Sigma_{i0b}^+ - \Sigma_{0a}^{-i} \Sigma_{i0b}^-] \\ & + \chi^a_b [\Sigma_{0a}^{+i} \tilde{E}_i^{+b} - \Sigma_{0a}^{-i} \tilde{E}_i^{-b}] + \tau_{ab} [\tilde{E}^{+ia} \tilde{E}_i^{+b} - \tilde{E}^{-ia} \tilde{E}_i^{-b}]. \end{aligned} \quad (53)$$

The lagrange multipliers $\tilde{\theta}$ and τ are symmetric while χ is trace free. ($\tilde{\theta}^{ab} =$

$4\psi^{0a0b}$, $\chi^a_b = 2\psi^{0acd}\epsilon_{bcd}$, and $\tau_{ab} = \frac{1}{4}\psi^{cdef}\epsilon_{acd}\epsilon_{bef}$). $\tilde{B}^{\pm ia} = \epsilon^{abc}F_{bc}^{\pm i}$ is the left/right $SU(2)$ magnetic field. One sees at once that the canonical coordinates are the $A_a^{\pm i}$ and their conjugate momenta, $\pm\tilde{E}_i^{\pm a}$. The field equations consist of constraints and expressions for the t derivatives of $A_a^{\pm i}$ and $\tilde{E}_i^{\pm a}$ as functions of both the spatial and time components of Σ^\pm , A^\pm , and the curvatures F^\pm appearing in the action. By our hypotheses on the fields these expressions must be finite, so the t evolution of $A_a^{\pm i}$ and \tilde{E}^\pm is continuous. It is clear from this point of view that there are no further, independent, requirements of continuity in t , since there are no further equations connecting the fields at different t .

Now let's see how the continuity conditions restrict transitions from the GR branch to other branches. Let's first consider transitions to the I_1 branch. Let B be the hypersurface separating the GR and I_1 phases and choose a piecewise differentiable slicing H_t so that H_0 coincides with B wherever B is differentiable. In particular this means that H_0 coincides with B whenever the fields are non-differentiable at B .⁹ If the $SO(4) = SU(2)_L \otimes SU(2)_R$ gauge is chosen so that $e_a^0 = 0$ the expressions (22) and (23) for Σ^\pm imply

$$\tilde{E}_i^{\pm a} = s^\pm e_i^a \det[e_b^j] \quad (54)$$

where e_i^a is the inverse of the 3×3 matrix e_b^j . From (54) follows the gauge invariant result that

$$\det[\tilde{E}_i^{\pm a}] = s^\pm \det[e_b^j]^2. \quad (55)$$

Thus $\det\tilde{E}^+ = \frac{s^+}{s^-} \det\tilde{E}^-$.

In the GR branch $s^+ = s^-$ while in the I_1 branch $s^+ = -s^-$, so to cross from one branch to another requires that $\det\tilde{E}^+$ or $\det\tilde{E}^-$ changes sign. The continuity of these determinants thus implies that one of them (and thus also the other) is zero on B . This in turn implies that $\det[e_a^i] \rightarrow 0$ as B is approached, so e^I_μ does *not* approach a non-degenerate limiting vierbein on B .

In the I_2 branch (in a suitable gauge) $\Sigma^+ = -\Sigma^-$ so $\det\tilde{E}^+ = -\det\tilde{E}^-$, as in the I_1 branch. Therefore, just as in the $GR \rightarrow I_1$ transition, $\det[e_\mu^I] = 0$ on B .

What about a transition to the I_3 branch? In this branch $\tilde{E}_i^{+a} = \tilde{E}_i^{-a}$ in an appropriate gauge, just as in the GR branch in the gauge in which $e_a^0 = 0$

⁹ It would of course be simpler if one could identify all of B with H_0 , but it is not clear to the author that all of B is even piecewise differentiable.

(recall (11)). Therefore it seems to be possible to make the transition with e^I_μ approaching a well defined and non-degenerate limiting vierbein as B is approached from the GR phase. We will see that this is in fact the case provided the curvature satisfies a constraint. Let's therefore suppose that $\det[e^I_\mu] \neq 0$ on an open subset U of B . In the I_3 phase $V = \frac{1}{2}(V^+ + V^-) = 0$, while in the GR phase it is $V = \det[e^I_\mu]\epsilon \neq 0$, so Σ^+ or Σ^- must be discontinuous on U . It follows that U is piecewise differentiable and coincides with H_0 .

In the I_3 phase $F^+ = F^-$ in the gauges in which $\Sigma^+ = \Sigma^-$. Thus the continuity of $A_a^{\pm i}$ implies that $F_{ab}^{+i} = F_{ab}^{-i}$ on U in the gauge $e^0_a = 0$. This is actually a sufficient condition for the existence of an I_3 branch solution matching the GR solution on B : provided this condition holds A^\pm and Σ^\pm constant in x^0 , with $A_0^{\pm i} = 0$ and $\Sigma_{i0a}^\pm = 0$, solve the I_3 field equations.

Reconstructing the $SO(4)$ curvature from F^\pm we find that the constraint $F^+ = F^-$ on the curvature is equivalent to $R^{0i}_{ab} = 0$ on U . It is thus possible to evolve from the GR phase to the I_3 phase without the vierbein becoming degenerate at the transition iff there exists a hypersurface in the GR solution on which $R^{0i}_{ab} = 0$. No such hypersurface exists in a generic solution to GR. The vacuum Einstein equations reduce the Riemann curvature to the Weyl curvature. $R^{0i}_{ab} = 0$ places five constraints on the Weyl curvature on the transition hypersurface. Two of these can be taken to be the vanishing of the two algebraically independent scalar densities¹⁰

$$a = R^{IJ}_{\alpha\beta} R_{IJ\gamma\delta} \epsilon^{\alpha\beta\gamma\delta} \quad (57)$$

¹⁰ The algebraic independence of a and b can be seen much more clearly when they are expressed in terms of the left and right handed Weyl curvature spinors. In a solution the left and right handed curvatures can be written as

$$F_{\mu\nu}^{\pm i} = \phi^{\pm ij} \Sigma_{j\mu\nu}^\pm, \quad (56)$$

where ϕ^+ and ϕ^- are traceless, symmetric 3×3 matrices. (*Outline proof:* $D^\pm \wedge \Sigma^\pm = 0 \Rightarrow D \wedge e = 0 \Rightarrow R^\alpha_{[\mu\nu\sigma]} = 0 \Rightarrow \psi^\sigma{}_\sigma{}^\beta = 0 \Rightarrow \Sigma_{\alpha\beta}^{+i} \psi^{\alpha\beta\gamma\delta} \Sigma_{\gamma\delta}^{-j} = 0$. This implies (56), and the tracelessness of ϕ^\pm follows straightforwardly from $\psi^{[\alpha\beta\gamma\delta]} = 0$). ϕ^\pm is the left (right) handed Weyl curvature spinor expressed as an $SO(3)$ tensor. Clearly our condition that the spatial components of the left and right handed curvatures be equal in the gauge in which $\tilde{E}^+ = \tilde{E}^-$ is equivalent to $\phi^+ = \phi^-$. In terms of the ϕ $a = (\text{tr}[\phi^{+2}] - \text{tr}[\phi^{-2}])\sqrt{g}$ and $b = (\det \phi^+ - \det \phi^-)\sqrt{g}$, modulo constant factors. From these forms it is easy to check algebraic independence.)

and

$$b = R_I^J{}_{\mu\alpha} R_J^K{}_{\beta\gamma} R_K^I{}_{\delta\nu} g^{\mu\nu} \epsilon^{\alpha\beta\gamma\delta}. \quad (58)$$

Using linear perturbation theory one can verify that any value of the Weyl curvature at a single point can be realized in a solution. That is to say, the consistency of the Einstein equations does not require any algebraic restriction on the Weyl curvature. Thus if a or b are zero at a point they will be perturbed to a non-zero value in a neighborhood of that point by a generic piecewise differentiable perturbation. A generic superposition of weak perturbations (i.e a generic small perturbation) will produce a and b that vanish only on some hypersurfaces. Now, the condition that $R^{0i}{}_{ab} = 0$ on the transition hypersurface requires that *both* a and b vanish on that hypersurface. But it is easy to see that a generic perturbation will separate the $a = 0$ and $b = 0$ hypersurfaces: Any perturbation can be combined with a diffeomorphism so that, at least in an open neighborhood, the $a = 0$ surface is preserved. The perturbations obtained in this way are in fact all the perturbations that preserve the $a = 0$ surface. Thus, since a and b are algebraically independent, these perturbations will generically make $b \neq 0$ at some point of the $a = 0$ surface, in fact they will make $b \neq 0$ everywhere on this surface except on some 2-surfaces. It follows that in a generic solution of GR there is no hypersurface on which $R^{0i}{}_{ab} = 0$ everywhere.

In terms of the canonical framework the preceeding results imply that complete left handed *and* right handed canonical data (A^\pm, \tilde{E}^\pm) corresponding to a non-degenerate solution of GR is not initial data for any I_1 or I_2 branch solution, and forms the initial data of an I_3 branch solution only in the exceptional case in which $F_{ab}^{+i} = F_{ab}^{-i}$ in the gauge in which $\tilde{E}^{+ia} = \tilde{E}^{-ia}$.¹¹

In GR the left handed part of the canonical data is sufficient to specify the whole solution. The conditions that define the GR branch (namely that the 4-volume density $2[\Sigma_{0a}^{+i} \tilde{E}_i^{+a} - \Sigma_{0a}^{-i} \tilde{E}_i^{-a}]$ not vanish and that $\det \tilde{E}^+$ and $\det \tilde{E}^-$ are non-zero with the same sign) and the constraints on the canonical variables and lagrange multipliers following from (53) imply that in a suitable gauge

$$\tilde{E}_i^{-a} = \tilde{E}_i^{+a} \quad (59)$$

$$A_a^{-i} = -A_a^{+i} + \epsilon_{jk}^i \omega^{jk}(\tilde{E}^+) \quad (60)$$

¹¹ In the exceptional case that the data is consistent with both a GR solution and an I_3 solution the branch must be selected by a suitable choice of the Lagrange multipliers Σ_{i0a}^\pm .

where ω_a^{jk} is the spatial spin connection compatible with the dreibein $e_i^a = \tilde{E}_i^{+a}/\sqrt{\det \tilde{E}^+}$. The constraints also determine the Lagrange multipliers A_0^- , Σ_{i0a}^- , θ , χ , and τ in terms of A_0^+ and Σ_{i0a}^+ in the same gauge, and thus allow the computation of the evolution of the canonical variables from the left handed Lagrange multipliers.

However, other right handed data can be chosen to match the given left handed data so that the development is an I_1 or I_3 branch solution. We may choose

$$\tilde{E}^- = -\tilde{E}^+ \quad (61)$$

$$A_a^{-i} = -A_a^{+i} + \epsilon_{jk}^i \omega_a^{jk}(\tilde{E}^+) \quad (62)$$

to obtain I_1 initial data, or

$$\tilde{E}^- = \tilde{E}^+ \quad (63)$$

$$A^- = A^+ \quad (64)$$

to get I_3 initial data. (Only in special cases is it possible to complete left handed data to form I_2 data, because $F^+ = -F^-$ requires that $-F^+$ is a curvature of some connection which is true only for certain special connections A^+).

4 A comment on quantizing the model

A fairly straightforward path integral quantization of the classical theory (18) defined by the principle “left-handed area = right-handed area” on a simplicial spacetime leads to the Barrett-Crane model.¹² Our study of the classical solutions however suggests a potential problem with such a path integral. It seems that semiclassically the path integral is completely dominated by histories near I_3 branch solutions, so that the quantum theory is a quantization of the I_3 branch only, and not the GR branch. The argument that leads to this conclusion runs as follows: Consider a formal path integral over the continuum fields, and take \tilde{E}^\pm as the boundary data to be kept fixed in the path integral. In a semiclassical situation the path integral should be dominated by histories near classical solutions. Suppose that the boundary data

¹² The author hopes to present the details in a forthcoming publication.

is compatible with a classical GR solution and, for notational convenience choose the gauge in which $\tilde{E}^+ = \tilde{E}^-$. In metric language we have fixed the 3-metric of the boundary and for the vierbein we have adopted the gauge in which e_0^μ is normal to the boundary). This boundary data will generically admit only one 4-diffeomorphism equivalence class of solutions to GR. But now consider I_3 solutions. Choose any Σ^+ such that $\Sigma_{ab}^{+i} = \frac{1}{2}\epsilon_{abc}\tilde{E}^{+ic}$ on the boundary. Whatever the boundary values are, almost all the spacetime fields Σ^+ will define an invertible matrix $m_{ij} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}\Sigma_{i\alpha\beta}^+\Sigma_{j\gamma\delta}^+$, and will thus define via (47) a unique connection A^+ satisfying the compatibility condition $D^+\wedge\Sigma^+ = 0$. By choosing $\Sigma^- = \Sigma^+$ and $A^- = A^+$ one obtains a complete I_3 solution matching the boundary data. Thus there is one solution for almost any Σ^+ satisfying the boundary data. This set of solutions is much larger than the 4-diffeo equivalence class of one solution. In GR too all the fields in a solution are determined by Σ^+ , up to an $SU(2)_R$ gauge freedom. For instance, the metric is given by Urbantke's formula [Urb83, Ben95]

$$\sqrt{g}g_{\mu\nu} = 8\epsilon^{\alpha\beta\gamma\delta}\Sigma_{\mu\alpha}^{+1}\Sigma_{\beta\gamma}^{+2}\Sigma_{\delta\nu}^{+3}. \quad (65)$$

But in GR only a very restricted set of Σ^+ histories correspond to solutions. If the 4-diffeo and $SU(2)_L \otimes SU(2)_R$ gauge freedom is fixed, for instance by the harmonic condition $\partial_\mu g^{\mu\nu} = 0$ on the inverse of the left-handed Urbantke metric (65), and $A_0^{+i} = 0$, $\Sigma_{ab}^{+i} = \Sigma_{ab}^{-i}$, then there will be only one GR solution matching the boundary data but an infinity of I_3 solutions. This suggests that in semiclassical situations the path integral is completely dominated by the I_3 branch solutions, so that the quantum theory defined by the path integral is *not* a quantization of GR.

5 Conclusion

The constraint “left-handed area = right-handed area” leads to a nice geometrical, and intuitively accessible, prescription for obtaining general relativity from BF theory. However, the resulting theory has several phases, only one of which reproduces GR. Moreover, some of the other phases contain in a sense many more solutions. GR is in this theory a sort of degenerate phase, analogous to unstable equilibria in mechanics, which classically can persist forever provided the initial data are taken from a special set of measure zero. Thus one has to take care when building a quantum theory of GR from BF

theory and “left-handed area = right-handed area” in order that the resulting theory is not simply the quantum theory of one of the other phases.

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